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Dual solutions of the magnetogasdynamic boundary-layer equations

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This paper extends the work of Wilson (1964) to include the effect of compressibility in the recurrence of dual solutions in the flow in the boundary layer on a semi-infinite, thermally insulated, flat plate placed at zero incidence to a uniform stream of electrically-conducting gas with an aligned magnetic field at large distances from the plate. Numerical integration of the boundary-layer equations has been performed for several values of the ratio, β, of the square of the Alfvén speed to the fluid speed in the undisturbed fluid, the conductivity parameter $\epsilon = 0{\cdot}1$ and ∞ and the square of the Mach number $M^2 = 0, \frac{1}{2}, 1, 2, 2{\cdot}5$, 4 and 5. The effect of compressibility is to increase the value of β for which a solution can exist such that the skin friction at the plate is greater than zero. Dual solutions are seen to occur for non-zero Mach number and all values of ϵ but no attempt here has been made to explain this phenomenon. An analytic argument indicates that no solutions of the equations exist if the skin friction at the plate is greater than zero and if the vorticity and current decay exponentially, the condition for which is

$$M^2 < 1, \quad \beta > 1/(1-M^2).$$

Nothing specific has been proved if this condition is not satisfied.

1. Introduction

The author in a previous paper (Ingham 1965), hereafter referred to as I, studied the flow of a viscous, electrically-conducting gas with variable conductivity and viscosity past a thin, semi-infinite, thermally insulated, flat plate in the presence of a magnetic field which is aligned with the free-stream velocity at large distances from the plate. The plate is at zero incidence to the main stream direction and a co-ordinate system is chosen in which the plate lies in the plane $y = 0, x > 0$. It was shown that for a highly conducting, almost inviscid gas with the Prandtl number P_r unity, the boundary layer on either side of the plate depends on the solution of the ordinary differential equations

$$d(\omega^2\nu f'')/d\eta + ff'' = \beta g(d/d\eta)(\omega g'), \tag{1.1a}$$

$$d(\omega g')/d\eta = \epsilon(\sigma/\omega)(f'g - fg'), \tag{1.1b}$$

$$\omega(1+M^2\tfrac{1}{2}(\gamma-1)(1-\tfrac{1}{4}f'^2)) = (1+\tfrac{1}{2}M^2\gamma\beta(1-\tfrac{1}{4}\omega^2g'^2)), \tag{1.1c}$$

subject to the boundary conditions

$$\left.\begin{aligned} f(0) = f'(0) = g(0) = 0,\\ f'(\infty) = g'(\infty) = 2.\end{aligned}\right\} \tag{1.2}$$

In these equations the primes denote differentiation with respect to the variable

$$\eta = \tfrac{1}{2}Y/\sqrt{x}, \tag{1.3}$$

where

$$Y = \int_0^y \rho\, dy, \tag{1.4}$$

and the suffix ∞ labels quantities evaluated in the main stream. The quantities ν, σ and ρ are the kinematic viscosity, electrical conductivity and density of the gas, made non-dimensional by using their main stream values and x and y are non-dimensional with respect to ν_∞/U_∞.

The functions f, g, and ω are associated with the velocity, magnetic field and the density, respectively, by

$$\psi = U_\infty x^{\frac{1}{2}} f(\eta), \quad A = H_\infty x^{\frac{1}{2}} g(\eta), \quad \rho = \rho_\infty \omega(\eta), \tag{1.5}$$

where ψ and A are the stream function and magnetic stream function, respectively, defined by

$$\left.\begin{aligned} \rho\mathbf{q} &= (\rho u, \rho v, 0) = \nabla \times (\psi \mathbf{k}), \\ \mathbf{H} &= (H_x, H_y, 0) = \nabla \times (A\mathbf{k}), \end{aligned}\right\} \tag{1.6}$$

where $\mathbf{k}$ is a unit vector in the z-direction.

The parameters occurring in equation (1.1) are

$$\beta = \mu H_\infty^2/(4\pi\rho_\infty U_\infty^2), \quad \epsilon = 4\pi\sigma_\infty \mu\nu_\infty, \quad M, \quad \gamma, \tag{1.7}$$

where β is the square of the ratio of the Alfvén speed to the fluid speed in the undisturbed fluid, ϵ is the ratio of the viscous and magnetic diffusivities, M is the Mach number, γ is the ratio of the specific heats of the gas and is taken to be 1·4, its value for air, and μ is the permeability of the gas and is assumed to be constant throughout.

In I numerical results were given for $\epsilon = 0{\cdot}01$ and $0{\cdot}1$, $\beta = 0$, $0{\cdot}3$ and $0{\cdot}5$, $\gamma = 1{\cdot}4$ and $M^2 = 0, \frac{1}{2}, 1, 2, 5, 10, 25$ and 100 taking σ/ω and $\omega^2\nu$ for air to vary as

$$\left.\begin{aligned} \omega^2\nu &= h^{-0\cdot35} = [1 + M^2\tfrac{1}{2}(\gamma-1)(1-\tfrac{1}{4}f'^2)]^{-0\cdot35}, \\ \sigma/\omega &= h^{5\cdot14} = [1 + M^2\tfrac{1}{2}(\gamma-1)(1-\tfrac{1}{4}f'^2)]^{5\cdot14}, \end{aligned}\right\} \tag{1.8}$$

(see Bush (1960)), where h is the specific enthalpy of the gas. In writing down these expressions the pressure dependence has been ignored since it has little influence on the description of the flow.

When $M = 0$ the equations (1.1) reduce to the well-known Greenspan–Carrier equations for an incompressible electrically-conducting fluid with constant properties namely

$$f''' + ff'' = \beta g g'', \tag{1.8a}$$

$$g'' = \epsilon(f'g - fg'), \tag{1.8b}$$

and, of course,

$$\omega = 1. \tag{1.8c}$$

Several authors have discussed the solution of these equations subject to the boundary condition (1.2). In particular, Greenspan & Carrier (1959) have given several approximate analytic and numerical solutions and Glauert (1961) obtained series solutions which are reliable for $\epsilon < 0{\cdot}001$ and $\epsilon > 10$ on the assumption that $1-\beta$ was not small; it was this work which was generalized in I for compressible flow past a thermally insulated plate. Reuter & Stewartson (1961) have shown that when $\beta > 1$ there are no solutions of the boundary-layer equations (1.8) subject to the boundary conditions (1.2) such that the skin friction evaluated at the plate is greater than zero (i.e. $f''(0) > 0$). Also Wilson (1964) and Stewartson & Wilson (1964) looked at the nature of the solution of the Greenspan–Carrier equations near $\beta = 1$. They found that for $\epsilon > 1$ a unique solution is possible as $\beta \to 1$, but for $\epsilon < 1$ there exists a β_0 such that in the range $\beta_0 < \beta < 1$ there are no solutions of the differential equations and for $\beta < \beta_0$ the solutions are not unique. They were unable to find a physical explanation of this phenomena. For $\beta > 1$, i.e. the Alfvén speed is greater than the fluid speed, disturbances can penetrate upstream of the plate so that the whole formulation of the problem in terms of the boundary-layer equations breaks down. In compressible flow (i.e. $M \neq 0$) this argument breaks down since there are now more than two characteristic speeds and an upstream wake will occur when the speed at which vorticity and current is carried is greater than the fluid speed. Fan (1964) showed that only one of the diffusion-type operators changes in character and leads to reversed diffusion when $(1+M^2\beta-\beta)$ changes its sign.

The present work was undertaken: (i) to see the effect of compressibility on the critical values of β for which there is no solution of equations (1.1); (ii) to investigate the possibility of dual solutions.

In regard to (i) Sears (1960) using the Fredrichs pulse diagram found that for an *inviscid*, compressible fluid of finite electrical conductivity, the flow and magnetic field both being aligned with the plate, the transition from a downstream inviscid wake to one upstream, and vice versa, takes place at flow conditions for which $1+M^2\beta-\beta = 0$. Resler & McCune (1960) also predicted this result. Fan (1964) found that this result is unaffected by viscosity but his analysis was greatly simplified by making an Oseen type of approximation for both the magnetic and flow fields, together with the magnetogasdynamic boundary-layer approximation; although the resulting equations are linear, the velocity, magnetic and temperature fields are still coupled.

2. Non-existence of solution

In I it was shown that for ϵ arbitrary and η large,

$$f \sim 2(\eta-a), \quad g \sim 2(\eta-a), \quad \omega \sim 1, \tag{2.1}$$

where a is some constant, and also

$$\left.\begin{aligned} &f'' \sim b\exp(-c(\eta-a)^2), \quad g'' \sim d\exp(-c(\eta-a)^2),\\ &\omega' \sim h\exp(-c(\eta-a)^2), \end{aligned}\right\} \tag{2.2}$$

where b, d and h are constants and c is the smaller root of the quadratic equation

$$c^2-c((\epsilon+1)+M^2\gamma\beta\epsilon)+\epsilon(1-\beta+M^2\beta) = 0. \tag{2.3}$$

Both roots of equation (2.3) are positive if

$$M^2 < 1, \quad \beta < 1/1-M^2 \tag{2.4}$$

or $M^2 \geqslant 1$ for all β, where γ has been taken to be always greater than unity. If condition (2.4) is not satisfied then this would imply, for large η, that f'', g'' and ω' increase exponentially with η, which is impossible.

The result is in full agreement with the linearized theory, which should be expected, since effectively equations (1.1) have been linearized. When $M = 0$ the condition (2.4) reduces to that obtained by Glauert from which he concluded that the asymptotic forms of the type (2.2) cannot exist for $\beta > 1$ since if they did, it would imply, for large η, that f'' and g'' increase exponentially with η which is impossible. Thus application of the 'Glauert' type argument to magnetogasdynamic flow *indicates* that difficulties may be encountered when $M^2 < 1$, $\beta > 1/(1-M^2)$, but when $M^2 < 1, \beta < 1/(1-M^2)$ and $M^2 \geqslant 1$ for all β, one is not entitled to say that, since both roots of (2.3) are positive, solutions exist.

A more rigorous proof that $\beta = 1$ is critical was given by Reuter & Stewartson and their method can easily be extended to the compressible case but with the conductivity infinite. The proof is omitted as it does not add appreciably to the Glauert argument and is only a modest extension of the Stewartson–Reuter argument.

3. Duality of solutions

A generalization of the analytic work done by Stewartson & Wilson has not been attempted but some numerical results have been obtained by the method, as described in I. Since the flow depends on five parameters, P_r, γ, ϵ, β and M^2 and is a two-point boundary-value problem reducing to a one-point boundary problem if $M = 0$, a complete solution of the equations would be too lengthy.

In the incompressible case, dual solutions occur if $\epsilon < 1$ but not if $\epsilon > 1$ and therefore the numerical results are limited to the two cases of $\epsilon = 0{\cdot}1$ and ∞. The nature of the dual solutions is investigated for $M^2 = 0, \frac{1}{2}, 1, 2, 2\frac{1}{2}, 4$ and 5, especially near the critical value of β above which no solutions can be obtained for a fixed value of M. The numerical results were checked by three methods:

(i) The step length at each step in the integration was decreased until the predetermined truncation error in any variable calculated was obtained. Varying the truncation error from 10^{-6} to 10^{-8} brought little change in the final result.

(ii) Having found the values for $f''(0)$ and $g'(0)$ for a particular ϵ, β and M the computed values at large η were put back on the computer and the integration back to $\eta = 0$ was performed, and again there was no significant difference between the starting values of $f''(0)$ and $g'(0)$.

(iii) The asymptotic solution for large η as given by (2.1) and (2.2) was compared with numerical results and for a wide range of η the difference between the numerical and asymptotic solutions is negligible.

4. Discussion of results

When $\epsilon = 0{\cdot}1$ the smaller value of the two solutions for each $\omega^2(0)\,\nu(0) f''(0)$ and $\omega(0)\,g'(0)$ are difficult to obtain numerically and therefore the results are restricted to values of $M^2 = 0, \frac{1}{2}, 1, 2$ and $2{\cdot}5$. For $M^2 = 4$ and 5 some dual solutions have been obtained, but since these do not cover a sufficiently large range of values of β they are not presented in full.

The skin friction at the plate, τ_ω, is given by

$$\tau_\omega = \tfrac{1}{4}(\rho_\infty U_\infty^2/x^{\frac{1}{2}})\,\omega^2(0)\,\nu(0)f''(0),$$

and the variation of $\omega^2(0)\,\nu(0)f''(0)$ with β for $M^2 = 0, \frac{1}{2}, 1, 2, 2{\cdot}5, 4$ and 5 with $\epsilon = 0{\cdot}1$ and ∞ are shown in figures 1 and 2 respectively. Figure 1 shows that dual

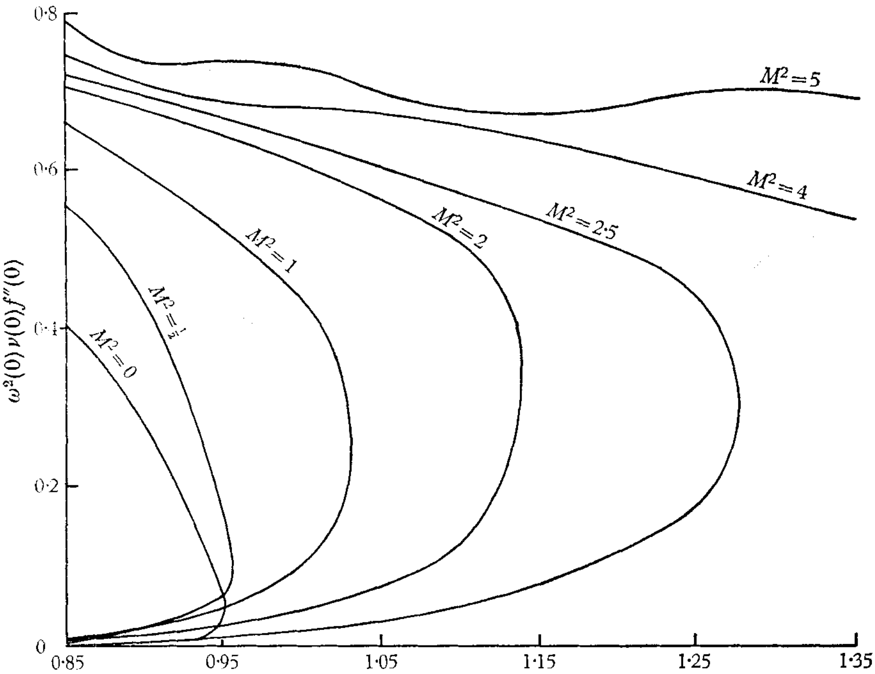

FIGURE 1. Graphs of $\omega^2(0)\,\nu(0)f''(0)$ against β for $\epsilon = 0{\cdot}1$.

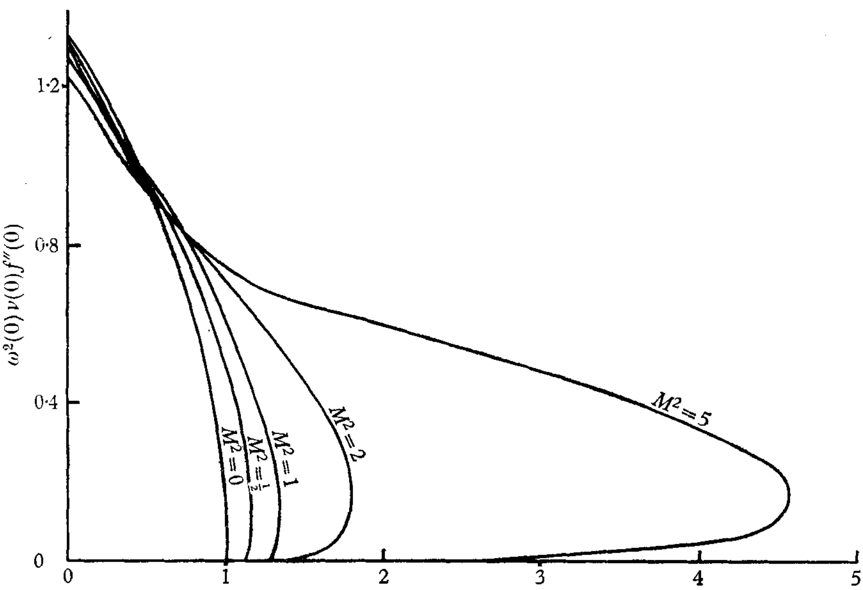

FIGURE 2. Graphs of $\omega^2(0)\,\nu(0)f''(0)$ against β for $\epsilon = \infty$.

solutions occur for all Mach numbers considered and, unlike the case for incompressible flow, dual solutions occur for $\beta > 1$ provided that M^2 is greater than some value between 0·5 and 1. Figure 2 shows that for infinite conductivity the critical value, β_0 say, above which no solution of the equations can be obtained, increases with Mach number. Two other interesting features are that: (i) dual solutions seem to appear for any finite value of the Mach number and $\epsilon = 0{\cdot}1$ and ∞; (ii) as the Mach number increases, for $\epsilon = 0{\cdot}1$ or ∞, β_0 increases.

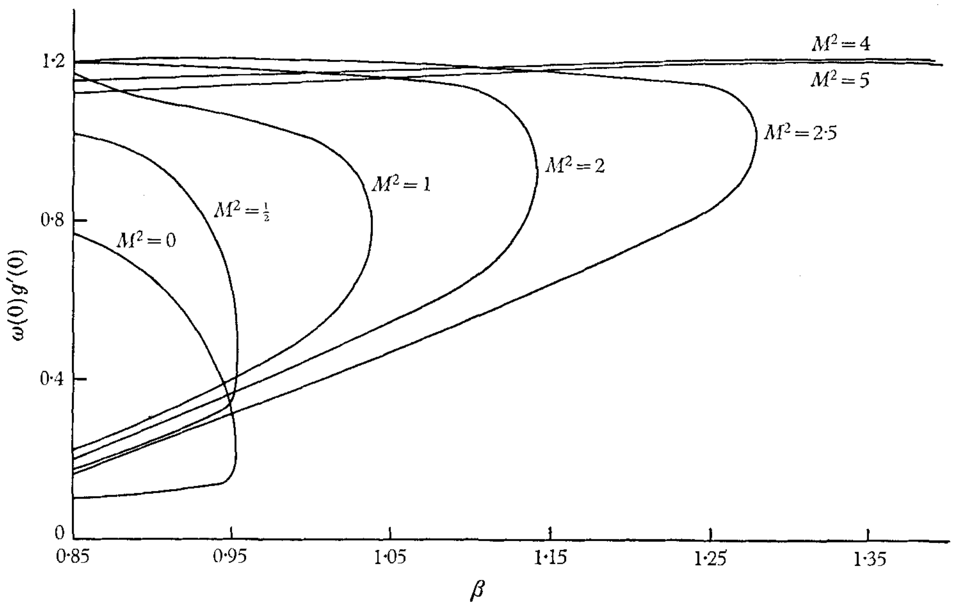

FIGURE 3. Graphs of $\omega(0)g'(0)$ against β for $\epsilon = 0{\cdot}1$.

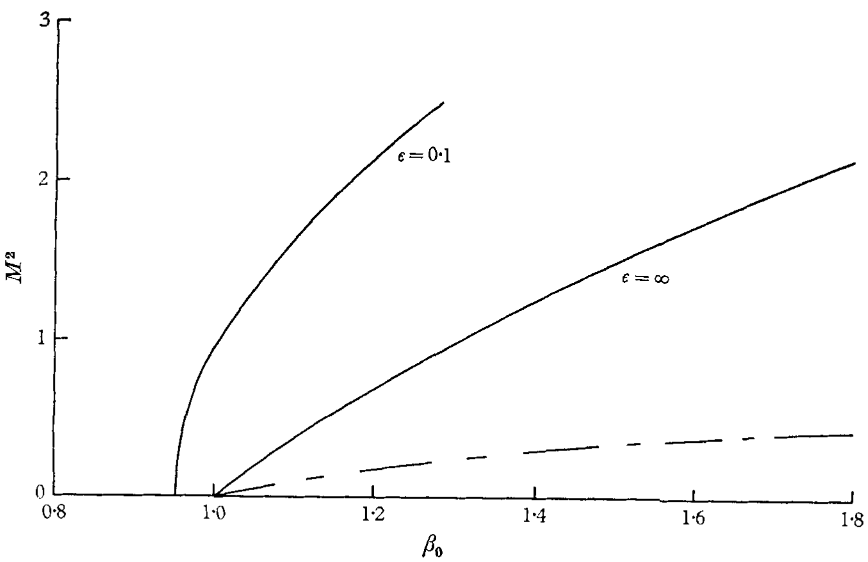

FIGURE 4. Graph of β_0 against M^2 for $\epsilon = 0{\cdot}1$ and ∞.
— – — – —, analytic upper bound for $\epsilon = \infty$.

The tangential component of the magnetic field is given by

$$H_x = \tfrac{1}{2} H_\infty \omega g',$$

and its variation with β at the plate for $M^2 = 0, \frac{1}{2}, 1, 2, 2\frac{1}{2}, 4$ and 5 with $\epsilon = 0{\cdot}1$ is shown in figure 3.

Figures 1–3 confirm the analytic result, namely that no solutions of the equations exist if $M^2 < 1$ and $\beta > 1/(1-M^2)$. It can be seen from figure 4 that the calculated values of β_0 are all less than $1/(1-M^2)$ for each value of $M < 1$, and for $M \geqslant 1$, β_0 is always finite and the difference between the calculated value of β_0 and the predicted upper bound is seen to increase with increasing Mach number. This is probably due to the limited applicability of the linearized procedure adopted, in particular it does not detect the rapidly changing conductivity.

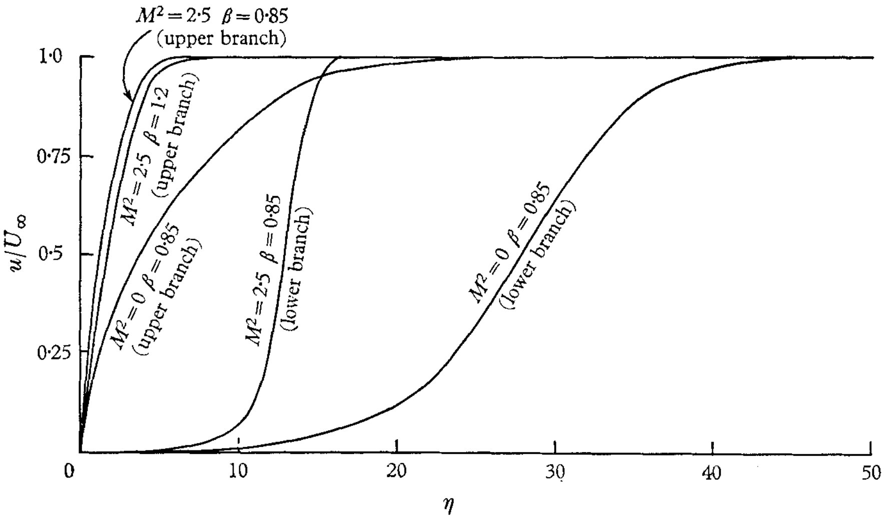

FIGURE 5. Profiles of the velocity component u for $M^2 = 0$ and $2{\cdot}5$, $\beta = 0{\cdot}85$ (upper and lower branches) and $M^2 = 2{\cdot}5$, $\beta = 1{\cdot}2$ (upper branch) in all cases for $\epsilon = 0{\cdot}1$.

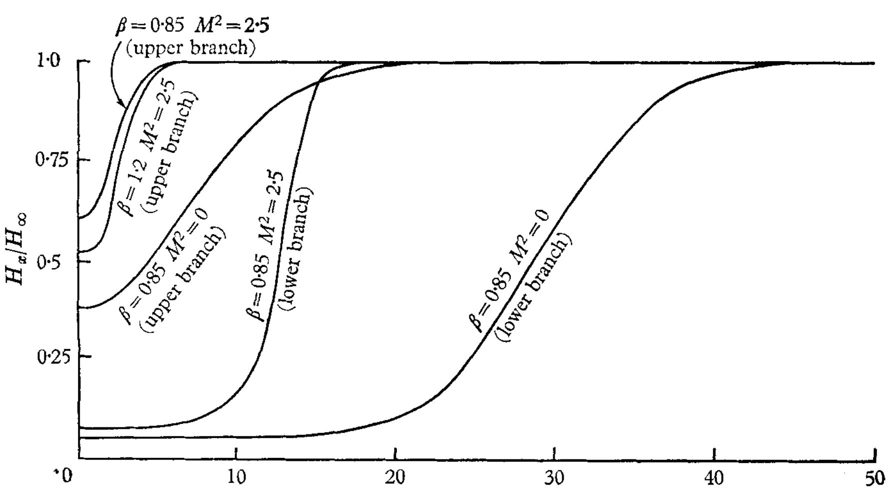

FIGURE 6. Profiles of magnetic field component H_x for $M^2 = 0$ and $2{\cdot}5$, $\beta = 0{\cdot}85$ (upper and lower branches) and $M^2 = 2{\cdot}5$, $\beta = 1{\cdot}2$ (upper branch) in all cases for $\epsilon = 0{\cdot}1$.

A region in which no solutions exist was also found in the incompressible case when $\epsilon < 1$. Stewartson & Wilson in their paper gave no physical explanation of this phenomenon and merely stated that 'the physical explanation of the non-existence, if $\beta_0 < \beta < 1$, is not clear because there is still no upstream propagation of small disturbances possible'. But in the boundary layer the *local* value of β [i.e. $(H_\infty^2/4\pi\rho_\infty U_\infty^2)(H_x^2/u^2)$] becomes very large near the plate where $u = 0$ and H_x is finite (except when the conductivity is infinite), and therefore it would seem that upstream propagation of small disturbances might be possible within the

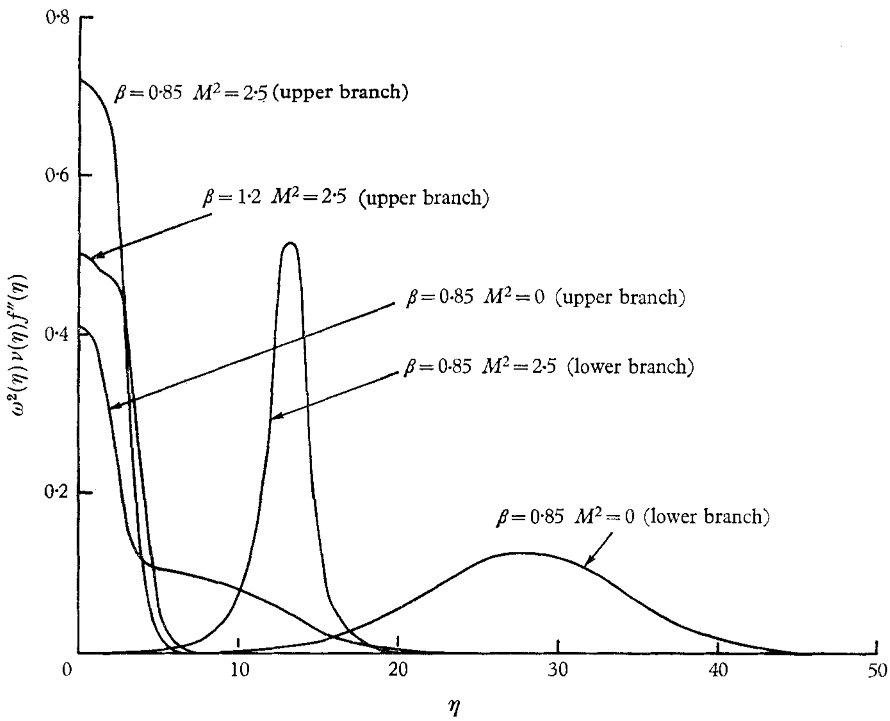

FIGURE 7. Profiles of the shear stress $\omega^2(\eta)\nu(\eta)f''(\eta)$ for $M^2 = 0$ and 2·5, $\beta = 0·85$ (upper and lower branches) and $M^2 = 2·5$, $\beta = 1·2$ (upper branch) in all cases for $\epsilon = 0·1$.

boundary layer itself. A closer examination of this effect might shed light on the reason for the non-existence of the solution if $\epsilon < 1$ and $\beta_0 < \beta < 1$. Obviously the reason given can also be applied to the compressible fluid.

One particularly interesting feature of the results when compressibility is included is the fact that the dual solutions appear for all values of the Mach number for which computations have been done for both $\epsilon = 0·1$ *and* ∞; this is not the case for the incompressible fluid ($M = 0$), where dual solutions appear only for $\epsilon < 1$. This suggests therefore that taking the value of $\epsilon = 1$ as critical is rather special, and peculiar only to the case $M = 0$.

For values of β where dual solutions exist, one might presume that the solution giving the larger value of the skin friction is appropriate since this solution covers all values of $\beta < \beta_0$ in a continuous manner. Here again a closer study of upstream influence is necessary.

Figures 5–7 show the profiles of the velocity (parallel to the plate the component of velocity is $u = \frac{1}{2}U_\infty f'(\eta)$), the magnetic-field component parallel to the plate and the variation of the shear stress across the boundary layer, respectively, for $\epsilon = 0{\cdot}1$, $M^2 = 0$ and $2{\cdot}5$ and $\beta = 0{\cdot}85$ (upper and lower branch) and $\beta = 1{\cdot}2$ (upper branch). It is observed that:

(i) The width of the boundary layer is decreased by increasing the Mach number for $\beta = 0{\cdot}85$ (upper and lower branch), illustrating the influence of compressibility on the magnetic Reynolds number.

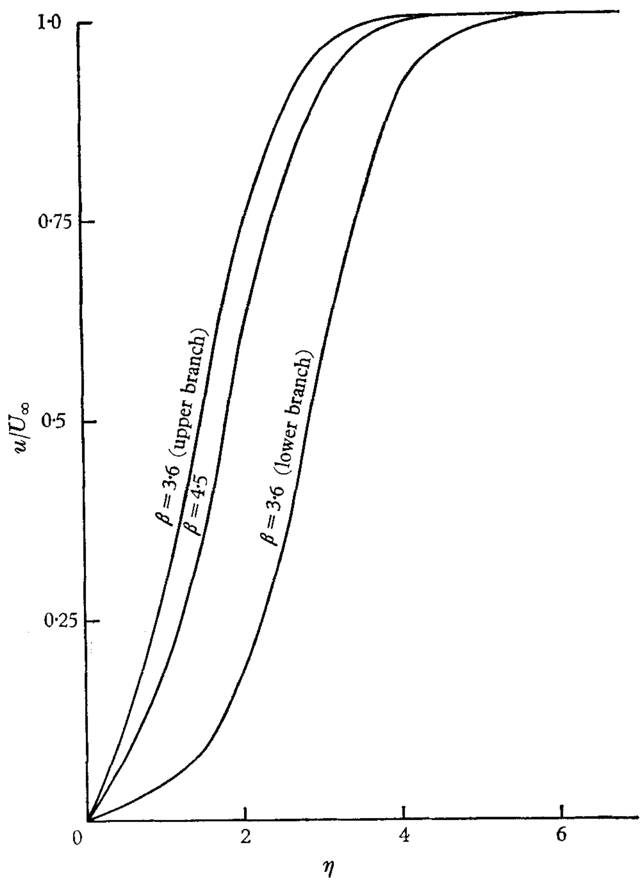

FIGURE 8. Profiles of the velocity component u for $\epsilon = \infty$, $M^2 = 5$, $\beta = 3{\cdot}6$ (upper and lower branches) and $4{\cdot}5$ (upper branch).

(ii) The peak of the curve in figure 7 for $\beta = 0{\cdot}85$ on the lower branch is much larger for $M^2 = 2{\cdot}5$ than the corresponding one for $M^2 = 0$.

(iii) For any given Mach number, as β increases the width of the boundary layer increases when the greater of the dual solutions is selected and decreases for the lower solutions.

Figures 8 and 9 show the velocity profiles and the variation of the shear stress across the boundary layer for infinite conductivity. It is observed that the comment (iii) applies here.

It can be seen from figures 7 and 9 that the solution corresponding to the lower value of the skin friction contains a region near the plate in which $d(\omega^2 \nu f'')/d\eta$ is greater than zero. In this region it is seen from equation (1.1 *a*) that the magnetic terms slightly dominate the convection terms which again is probably because the local value of β in this region is greater than unity. When $\epsilon = 0{\cdot}1$, figure 7 shows that $\omega^2 \nu f''$ is very small near the plate and therefore the boundary layer can be considered to consist of three regions—a magnetic, viscous and inviscid boundary layer.

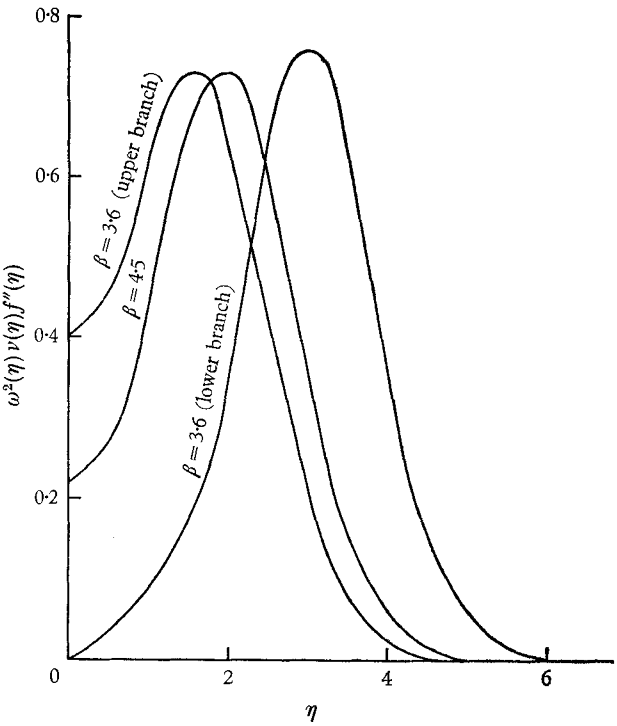

FIGURE 9. Profiles of the shear stress $\omega^2(\eta)\nu(\eta)f''(\eta)$ for $\epsilon = \infty$, $M^2 = 5$, $\beta = 3{\cdot}6$ (upper and lower branches) and 4·5 (upper branch).

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